

Electrodynamic spherical harmonic

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Abstract

Electrodynamic spherical harmonic is a second rank tensor in three-dimensional space. It allows to separate the radial and angle variables in vector solutions of Maxwell's equations. Using the orthonormalization for electrodynamic spherical harmonic, a boundary problem on a sphere can be easily solved.

1 Introduction

In this paper we introduce new function — electrodynamic spherical harmonic. It is represented as a second rank tensor in three-dimensional space. But the function differs from tensor spherical harmonic [1–5]. By its definition the electrodynamic spherical function possesses a number of properties of usual scalar and vector harmonics and includes them as component parts. Why *electrodynamic* spherical harmonic? The word “electrodynamic” implies that the function is applied for solution of *vector* field problems. We use it in electrodynamics, however one can apply the function for description of spin fields in quantum field theory.

Electrodynamic spherical harmonic is not a simple designation of the well-known functions. It satisfies the Maxwell equations and describes the angular dependence of vector fields. The introduced function separates the variables (radial coordinate and angles) in the fields. Moreover, the notation of the fields in terms of electrodynamic spherical harmonic noticeably simplifies the solution of a boundary problem on spherical interface. Just application of

the orthonormalization condition allows to find the coefficients of spherical function expansion (e.g. scattering field amplitudes).

2 Scalar and vector spherical harmonics

Hamilton's operator ∇ in three-dimensional space contains the derivatives on three coordinates. In spherical coordinates (r, θ, φ) the derivatives on radial coordinate and angles can be separated. This is achieved by representing the unit tensor $\mathbf{1}$ in 3D space as the superposition of two projection operators:

$$\nabla = \mathbf{1}\nabla = \left(\frac{\mathbf{r} \otimes \mathbf{r}}{r^2} - \frac{\mathbf{r}^{\times 2}}{r^2} \right) \nabla = \frac{1}{r^2} \mathbf{r}(\mathbf{r}\nabla) - \frac{1}{r^2} \mathbf{r}^{\times}(\mathbf{r}^{\times}\nabla),$$

where $\mathbf{r} \otimes \mathbf{r}/r^2$ is the projector onto the direction $\mathbf{e}_r = \mathbf{r}/r$, $-\mathbf{r}^{\times 2}/r^2$ is the projector onto the plane orthogonal to the unit vector \mathbf{e}_r . Tensor \mathbf{r}^{\times} dual to the vector \mathbf{r} gives the well-known vector product when acting on a vector \mathbf{a} : $\mathbf{r}^{\times}\mathbf{a} = \mathbf{r} \times \mathbf{a}$ and $\mathbf{a}\mathbf{r}^{\times} = \mathbf{a} \times \mathbf{r}$ [6]. Introducing the vector differential operator

$$\mathbf{L} = \frac{1}{i} \mathbf{r} \times \nabla \quad (1)$$

the equation (??) is rewritten as follows

$$\nabla = \frac{\mathbf{r}}{r} \frac{\partial}{\partial r} - \frac{i}{r^2} \mathbf{r} \times \mathbf{L}. \quad (2)$$

Vector \mathbf{L} is called orbital angular momentum operator in quantum mechanics, because it is presented as vector product of radius vector \mathbf{r} and momentum $\mathbf{p} = -i\nabla$ operators. \mathbf{L} includes only derivatives on the angles θ and φ . Using the definition of \mathbf{L} one can derive the Laplace operator

$$\Delta = \nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) - \frac{\mathbf{L}^2}{r^2} \quad (3)$$

and the following properties:

$$\mathbf{r}\mathbf{L} = 0, \quad (\mathbf{L}\mathbf{r}) = 0, \quad \mathbf{L}^2\mathbf{L} = \mathbf{L}\mathbf{L}^2, \quad \mathbf{L} \times \mathbf{L} = i\mathbf{L}. \quad (4)$$

Scalar spherical harmonic $Y_{lm}(\theta, \varphi)$ is defined as the eigenfunction of the operator \mathbf{L}^2 :

$$\mathbf{L}^2 Y_{lm} = l(l+1) Y_{lm}, \quad (5)$$

where l and m are integer numbers. Number m is the eigenvalue of the operator of projection of angular momentum onto the axis z , L_z :

$$L_z Y_{lm} = m Y_{lm}. \quad (6)$$

Spherical harmonics Y_{lm} are orthogonal and normalized by the unit:

$$\int_0^\pi \int_0^{2\pi} Y_{l'm'}^*(\theta, \varphi) Y_{lm}(\theta, \varphi) \sin \theta d\theta d\varphi = \delta_{l'l} \delta_{m'm}. \quad (7)$$

where the sign $*$ denotes the complex conjugate.

If we multiply equation (5) by the vector operator \mathbf{L} and take into account the commutation of \mathbf{L} and \mathbf{L}^2 , we will obtain that vector $\mathbf{L}Y_{lm}$ satisfies the same equation (5), too. The quantity defined as

$$\mathbf{X}_{lm} = \frac{1}{\sqrt{l(l+1)}} \mathbf{L}Y_{lm} \quad (8)$$

is called vector spherical harmonic. The coefficient before $\mathbf{L}Y_{lm}$ is chosen so that the orthonormalization condition is of the form

$$\int_0^\pi \int_0^{2\pi} \mathbf{X}_{l'm'}^*(\theta, \varphi) \mathbf{X}_{lm}(\theta, \varphi) \sin \theta d\theta d\varphi = \delta_{l'l} \delta_{m'm}. \quad (9)$$

From the self-conjugacy of the angular momentum operator \mathbf{L} and properties (4) the orthogonality

$$\begin{aligned} \int_0^\pi \int_0^{2\pi} \mathbf{e}_r (\mathbf{X}_{l'm'}^* \times \mathbf{X}_{lm}) \sin \theta d\theta d\varphi &= \int_0^\pi \int_0^{2\pi} \frac{Y_{l'm'}^* \mathbf{e}_r (\mathbf{L} \times \mathbf{L}) Y_{lm}}{\sqrt{l(l+1)l'(l'+1)}} \sin \theta d\theta d\varphi \\ &= \int_0^\pi \int_0^{2\pi} \frac{i Y_{l'm'}^* \mathbf{e}_r \mathbf{L} Y_{lm}}{\sqrt{l(l+1)l'(l'+1)}} \sin \theta d\theta d\varphi \quad (10) \end{aligned}$$

follows. Below we give some properties of vector spherical harmonics:

$$\mathbf{L} \mathbf{X}_{lm} = \sqrt{l(l+1)} Y_{lm}, \quad \mathbf{L}(\mathbf{e}_r \times \mathbf{X}_{lm}) = 0. \quad (11)$$

Scalar (vector) spherical harmonics satisfy the scalar (vector) equation for eigenfunctions of the squared orbital angular momentum operator \mathbf{L}^2 . The orthogonality condition for the scalar (7) and vector (9) spherical harmonics have the same form. Therefore, one can hope to combine them into one mathematical object.

3 Electrodynamic spherical harmonic: definition and properties

We define an electrodynamic spherical harmonic as a second rank tensor in three-dimensional space

$$F_{lm} = Y_{lm} \mathbf{e}_r \otimes \mathbf{e}_r + \mathbf{X}_{lm} \otimes \mathbf{e}_\theta + (\mathbf{e}_r \times \mathbf{X}_{lm}) \otimes \mathbf{e}_\varphi. \quad (12)$$

The first term of (12) determines the longitudinal part of the tensor. It is calculated by means of the scalar spherical function. The last two summands of (12) fix the transverse solution, in the plane (θ, φ) perpendicular to the direction \mathbf{e}_r . It includes vector spherical harmonics. The left and right vectors in dyads form two sets of orthogonal vectors: $(Y_{lm} \mathbf{e}_r, \mathbf{X}_{lm}, \mathbf{e}_r \times \mathbf{X}_{lm})$ and $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\varphi)$.

3.1 Orthonormalization

Multiplying the electrodynamic spherical harmonic by the Hermitian conjugate F_{lm}^+ we get to

$$F_{l'm'}^+ F_{lm} = Y_{l'm'}^* Y_{lm} \mathbf{e}_r \otimes \mathbf{e}_r + \mathbf{X}_{l'm'}^* \mathbf{X}_{lm} (\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \mathbf{e}_\varphi \otimes \mathbf{e}_\varphi) + (\mathbf{e}_r (\mathbf{X}_{l'm'}^* \times \mathbf{X}_{lm})) \mathbf{e}_r^\times. \quad (13)$$

The quantity before each dyad is orthogonal or normalized as (7), (9), or (10). Therefore, the orthonormalization condition for electrodynamic spherical harmonics is

$$\int_0^\pi \int_0^{2\pi} F_{lm}^+(\theta, \varphi) F_{lm}(\theta, \varphi) \sin \theta d\theta d\varphi = \mathbf{1} \delta_{l'l} \delta_{m'm}. \quad (14)$$

Here we consider the vectors \mathbf{e}_r , \mathbf{e}_θ , and \mathbf{e}_φ to be constant in dyads and dual tensor \mathbf{e}_r^\times . If tensors F_{lm} are the *solutions* of equations, we can always write these solutions in components. For each component of the tensor $F_{lm}^+ F_{lm}$ the orthonormalization is carried out. We will obtain the same, if the dependence of the orts \mathbf{e}_r , \mathbf{e}_θ , \mathbf{e}_φ on angles in (13) is omitted, i.e. they are regarded as constants. However, when substituting F_{lm} in equations, we should take into account the angular dependence of the orts.

3.2 Explicit form

Let us substitute the explicit expression for the vector operator \mathbf{L}

$$\mathbf{L} = -i\mathbf{e}_\varphi \frac{\partial}{\partial \theta} + i \frac{\mathbf{e}_\theta}{\sin \theta} \frac{\partial}{\partial \varphi} \quad (15)$$

into equation (12). Then the electrodynamic spherical harmonic is equal to

$$F_{lm} = \left[\mathbf{e}_r \otimes \mathbf{e}_r + \frac{i}{\sin \theta \sqrt{l(l+1)}} \left(I \frac{\partial}{\partial \varphi} - \mathbf{e}_r^\times \sin \theta \frac{\partial}{\partial \theta} \right) \right] Y_{lm}, \quad (16)$$

where $I = -\mathbf{e}_r^\times \mathbf{e}_r^\times = \mathbf{1} - \mathbf{e}_r \otimes \mathbf{e}_r$ is the projection operator onto the plane (θ, φ) . To calculate the derivatives one can replace them by means of operators L_z and $L_\pm = L_x \pm iL_y$ as

$$\frac{\partial}{\partial \varphi} = iL_z, \quad \frac{\partial}{\partial \theta} = \frac{1}{2}(e^{-i\varphi}L_+ - e^{i\varphi}L_-),$$

because their action on the scalar spherical harmonic is well-known:

$$L_+ Y_{lm} = \sqrt{(l-m)(l+m+1)} Y_{l,m+1}, \quad L_- Y_{lm} = \sqrt{(l+m)(l-m+1)} Y_{l,m-1}. \quad (17)$$

Hence, the electrodynamic spherical harmonic can be presented as follows

$$F_{lm} = \left[\mathbf{e}_r \otimes \mathbf{e}_r - \frac{1}{\sin \theta \sqrt{l(l+1)}} \left(IL_z + \mathbf{e}_r^\times \frac{i \sin \theta}{2} (e^{-i\varphi}L_+ - e^{i\varphi}L_-) \right) \right] Y_{lm}. \quad (18)$$

It is easy to exclude the operators from (18). The final formula for tensor F_{lm} contains scalar spherical harmonics as angle dependence. Unit vectors $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\varphi$ determine the structure of the tensor in three-dimensional space. F_{lm} is formed by three basic tensors: $\mathbf{e}_r \otimes \mathbf{e}_r$, \mathbf{e}_r^\times , and I . Hence, it commutes with each of these tensors.

3.3 Invariants

The first invariant of the electrodynamic spherical harmonic as tensor quantity is its trace. The trace of the tensor (12) equals

$$\text{tr}(F_{lm}) = Y_{lm} + 2(\mathbf{e}_\theta \mathbf{X}_{lm}). \quad (19)$$

The second invariant is determinant

$$\det(F_{lm}) = Y_{lm} \mathbf{X}_{lm}^2. \quad (20)$$

The third invariant of three-dimensional tensor F_{lm} is the trace of the adjoint tensor \overline{F}_{lm} . Adjoint tensor is defined by $\overline{F}_{lm} F_{lm} = F_{lm} \overline{F}_{lm} = \det(F_{lm}) \mathbf{1}$ and equals

$$\overline{F}_{lm} = \mathbf{X}_{lm}^2 \mathbf{e}_r \otimes \mathbf{e}_r + Y_{lm} \mathbf{e}_\theta \otimes \mathbf{X}_{lm} + Y_{lm} \mathbf{e}_\varphi \otimes \mathbf{e}_r \times \mathbf{X}_{lm}. \quad (21)$$

Further the trace is easily calculated:

$$\text{tr}(\overline{F}_{lm}) = \mathbf{X}_{lm}^2 + 2Y_{lm}(\mathbf{e}_\theta \mathbf{X}_{lm}). \quad (22)$$

Using these three invariants one can find other ones. For example, the trace of squared tensor is determined from equation $\text{tr}(F_{lm}^2) = (\text{tr}(F_{lm}))^2 - 2\text{tr}(\overline{F}_{lm})$.

3.4 Generalization of electrodynamic spherical harmonic

The main condition on electrodynamic spherical harmonic is the orthonormalization (14). There is more general form of the second rank tensor spherical harmonic satisfying this equation. It is

$$G_{lm} = Y_{lm} \mathbf{e}_r \otimes \mathbf{a} + \mathbf{X}_{lm} \otimes \mathbf{b} + (\mathbf{e}_r \times \mathbf{X}_{lm}) \otimes \mathbf{c}. \quad (23)$$

Unit vectors \mathbf{a} , \mathbf{b} , and \mathbf{c} form the orthogonal basis in three-dimensional space. In equation (12) we have assumed $\mathbf{a} = \mathbf{e}_r$, $\mathbf{b} = \mathbf{e}_\theta$, $\mathbf{c} = \mathbf{e}_\varphi$ because of the spherical symmetry of the function. If we will take $\mathbf{a} = \mathbf{e}_r$, $\mathbf{b} = \mathbf{e}_\theta$, $\mathbf{c} = -\mathbf{e}_\varphi$, then the electrodynamic spherical harmonic becomes more complex function in explicit form, however its invariants are simplified (e.g., the trace equals $\text{tr}(F_{lm}) = Y_{lm}$).

4 Solution of Maxwell's equations

In this section we will find the spherically symmetric solutions (i.e. electric \mathbf{E} and magnetic \mathbf{H} fields) of the Maxwell equations

$$\nabla \times \mathbf{E}(\mathbf{r}) = ik\mu \mathbf{H}(\mathbf{r}), \quad \nabla \times \mathbf{H}(\mathbf{r}) = -ik\varepsilon \mathbf{E}(\mathbf{r}) \quad (24)$$

for the monochromatic electromagnetic waves in isotropic medium with dielectric permittivity ε and magnetic permeability μ . The quantity $k = \omega/c$ is called wavenumber in vacuum, and ω is the wave frequency. Arbitrary time dependence can be obtained by using the linear superposition of monochromatic waves $\mathbf{E}(\mathbf{r}) \exp(-i\omega t)$.

We will search the solution in the form

$$\mathbf{E}(\mathbf{r}) = F_{lm}(\theta, \varphi) \mathbf{E}^l(r). \quad (25)$$

The components of the vector \mathbf{E}^l in spherical coordinates depend only on the radial coordinate r . The dependence on the angle coordinates presents only in the basis vectors. So, the vector \mathbf{E}^l looks like

$$\mathbf{E}^l(r) = E_r^l(r) \mathbf{e}_r + E_\theta^l(r) \mathbf{e}_\theta + E_\varphi^l(r) \mathbf{e}_\varphi. \quad (26)$$

Further we should calculate $\text{rot} \mathbf{E}$. By substituting Hamilton's operator (2) one obtains

$$\nabla \times \mathbf{E} = \mathbf{e}_r^\times \frac{\partial \mathbf{E}}{\partial r} - \frac{i}{r} \mathbf{L}(\mathbf{e}_r \overset{\downarrow}{\mathbf{E}}) + \frac{i}{r} \mathbf{e}_r (\mathbf{L} \mathbf{E}), \quad (27)$$

where the arrow \downarrow implies that operator \mathbf{L} acts only on vector \mathbf{E} , but not \mathbf{e}_r . Let us calculate each summand of equation (27) applying the solution (25). The first term is of the form

$$\mathbf{e}_r^\times \frac{\partial \mathbf{E}}{\partial r} = F_{lm} \mathbf{e}_r^\times \frac{\partial \mathbf{E}^l}{\partial r}, \quad (28)$$

where the commutation relation $[F_{lm}, \mathbf{e}_r^\times] = 0$ is taken into account. The second summand yields

$$\mathbf{L}(\mathbf{e}_r \overset{\downarrow}{\mathbf{E}}) = \mathbf{L}(\mathbf{e}_r \mathbf{E}) - \mathbf{L}(\overset{\downarrow}{\mathbf{e}}_r \mathbf{E}) = \mathbf{L}(Y_{lm} E_r^l) - \mathbf{L}(\overset{\downarrow}{\mathbf{e}}_r \mathbf{E}). \quad (29)$$

The quantity $\mathbf{L}(\overset{\downarrow}{\mathbf{e}}_r \mathbf{E})$ is easily calculated using the explicit expression (15) of the operator \mathbf{L} and the relationships $\partial \mathbf{e}_r / \partial \theta = \mathbf{e}_\theta$ and $\partial \mathbf{e}_r / \partial \varphi = \mathbf{e}_\varphi \sin \theta$:

$$\mathbf{L}(\overset{\downarrow}{\mathbf{e}}_r \mathbf{E}) = -i \mathbf{e}_r^\times \mathbf{E}. \quad (30)$$

So, we get the formula

$$\mathbf{L}(\mathbf{e}_r \overset{\downarrow}{\mathbf{E}}) = F_{lm} \left(\sqrt{l(l+1)} \mathbf{e}_\theta \otimes \mathbf{e}_r + i \mathbf{e}_r^\times \right) \mathbf{E}^l. \quad (31)$$

The third term in (27) can be rewritten using the equation (11):

$$\mathbf{e}_r(\mathbf{L}\mathbf{E}) = \mathbf{e}_r \left(E_r^l \mathbf{L}(Y_{lm} \mathbf{e}_r) + E_\theta^l \mathbf{L}\mathbf{X}_{lm} + E_\varphi^l \mathbf{L}(\mathbf{e}_r \times \mathbf{X}_{lm}) \right) = \sqrt{l(l+1)} F_{lm}(\mathbf{e}_r \otimes \mathbf{e}_\theta) \mathbf{E}^l. \quad (32)$$

Finally, the curl of the electric field vector \mathbf{E} equals

$$\nabla \times \mathbf{E} = F_{lm}(\theta, \varphi) \left(\mathbf{e}_r^\times \frac{\partial}{\partial r} + \frac{1}{r} \mathbf{e}_r^\times - \frac{i\sqrt{l(l+1)}}{r} \mathbf{e}_\varphi^\times \right) \mathbf{E}^l(r). \quad (33)$$

In expression (33) we have took into account the derivatives on the angle variables. Therefore, further the orts of spherical coordinates \mathbf{e}_r , \mathbf{e}_θ , and \mathbf{e}_φ can be considered as constants. The Maxwell equations are reduced to the set of ordinary differential equations

$$\begin{aligned} \mathbf{e}_r^\times \frac{d\mathbf{E}^l}{dr} + \frac{1}{r} \mathbf{e}_r^\times \mathbf{E}^l - \frac{i\sqrt{l(l+1)}}{r} \mathbf{e}_\varphi^\times \mathbf{E}^l &= ik\mu \mathbf{H}^l, \\ \mathbf{e}_r^\times \frac{d\mathbf{H}^l}{dr} + \frac{1}{r} \mathbf{e}_r^\times \mathbf{H}^l - \frac{i\sqrt{l(l+1)}}{r} \mathbf{e}_\varphi^\times \mathbf{H}^l &= -ik\varepsilon \mathbf{E}^l. \end{aligned} \quad (34)$$

Equations (34) allow to determine the radial dependence of the fields. Multiplying the set of equations (34) by the unit vector \mathbf{e}_r we can express the longitudinal components of the fields as follows

$$E_r^l = -\frac{\sqrt{l(l+1)}}{\varepsilon kr} H_\theta^l, \quad H_r^l = \frac{\sqrt{l(l+1)}}{\mu kr} E_\theta^l. \quad (35)$$

The tangential field components $\mathbf{E}_t^l = I\mathbf{E}^l = E_\theta^l \mathbf{e}_\theta + E_\varphi^l \mathbf{e}_\varphi$ and $\mathbf{H}_t^l = I\mathbf{H}^l$ are determined from the equations which can be written in matrix form:

$$\frac{d(r\mathbf{W})}{dr} = ikM(r\mathbf{W}), \quad (36)$$

where

$$\mathbf{W} = \begin{pmatrix} \mathbf{H}_t^l \\ \mathbf{E}_t^l \end{pmatrix}, \quad M = \begin{pmatrix} 0 & \varepsilon A \\ -\mu A & 0 \end{pmatrix}, \quad A = \mathbf{e}_r^\times - \frac{l(l+1)}{\varepsilon \mu k^2 r^2} \mathbf{e}_\varphi \otimes \mathbf{e}_\theta. \quad (37)$$

Equation (36) is satisfied for inhomogeneous media $\varepsilon(r)$, $\mu(r)$, too. Such matrix equation can be solved numerically for arbitrary medium, or analytically for homogeneous one. Let us find tangential field components \mathbf{W} when

$\varepsilon = \text{const}$ and $\mu = \text{const}$. The simplest way is to write the equation for projection $W_\theta = (H_\theta^l, E_\theta^l)$:

$$\frac{d^2(rW_\theta)}{dr^2} + \left(k^2\varepsilon\mu - \frac{l(l+1)}{r^2} \right) (rW_\theta) = 0, \quad (38)$$

The solutions of the equation (38) are well-known and can be presented as

$$W_\theta = f^{(1)} \begin{pmatrix} c_1 \\ c_1' \end{pmatrix} + f^{(2)} \begin{pmatrix} c_2 \\ c_2' \end{pmatrix} = (f^{(1)}\mathbf{c}_1 + f^{(2)}\mathbf{c}_2) \begin{pmatrix} \mathbf{e}_\theta \\ \mathbf{e}_\varphi \end{pmatrix}, \quad (39)$$

where \mathbf{c}_1 and \mathbf{c}_2 are constant vectors. The couples of independent solutions are spherical Bessel functions $f^{(1)} = j_l(nkr)$, $f^{(2)} = j_{-l-1}(nkr)$ or spherical Hankel functions of the first and second kind $f^{(1)} = h_l^{(1)}(nkr)$, $f^{(2)} = h_l^{(2)}(nkr)$. $n = \sqrt{\varepsilon\mu}$ is the refractive index. After determining the φ -projections of the fields as

$$W_\varphi = \frac{i}{kr\varepsilon\mu} \begin{pmatrix} 0 & -\varepsilon \\ \mu & 0 \end{pmatrix} \frac{d(rW_\theta)}{dr} \quad (40)$$

one can write the transverse vector field

$$\mathbf{W}(r) = \begin{pmatrix} \eta_1(r) & \eta_2(r) \\ \zeta_1(r) & \zeta_2(r) \end{pmatrix} \begin{pmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \end{pmatrix}, \quad (41)$$

where

$$\begin{aligned} \eta_{1,2} &= f^{(1,2)}\mathbf{e}_\theta \otimes \mathbf{e}_\theta - \frac{i}{\mu kr} \frac{d(rf^{(1,2)})}{dr} \mathbf{e}_\varphi \otimes \mathbf{e}_\varphi, \\ \zeta_{1,2} &= f^{(1,2)}\mathbf{e}_\theta \otimes \mathbf{e}_\varphi + \frac{i}{\varepsilon kr} \frac{d(rf^{(1,2)})}{dr} \mathbf{e}_\varphi \otimes \mathbf{e}_\theta. \end{aligned} \quad (42)$$

Tangential field vector \mathbf{W} plays an important part, because it is continuous on the spherical surface. That is why it can be applied for the study of electromagnetic wave diffraction by a sphere.

5 Conclusion

Thus, the general solution of the Maxwell equations is of the form

$$\begin{pmatrix} \mathbf{H}(\mathbf{r}) \\ \mathbf{E}(\mathbf{r}) \end{pmatrix} = \sum_{l=0}^{\infty} \sum_{m=-l}^l F_{lm}(\theta, \varphi) V^l(r) \begin{pmatrix} \eta_1^l(r) & \eta_2^l(r) \\ \zeta_1^l(r) & \zeta_2^l(r) \end{pmatrix} \begin{pmatrix} \mathbf{c}_1^l \\ \mathbf{c}_2^l \end{pmatrix}, \quad (43)$$

where V^l is the matrix that takes into account the longitudinal components of electric and magnetic fields. This matrix can be easily calculated from the equation (35). In each partial solution included into the general one (43) the radial and angle variables are separated. Using the orthonormalization (14) for the electrodynamic spherical harmonic F_{lm} , each partial wave can be easily singled out:

$$V^l(r) \begin{pmatrix} \eta_1^l(r) & \eta_2^l(r) \\ \zeta_1^l(r) & \zeta_2^l(r) \end{pmatrix} \begin{pmatrix} \mathbf{c}_1^l \\ \mathbf{c}_2^l \end{pmatrix} = \int_0^\pi \int_0^{2\pi} F_{lm}^+(\theta', \varphi') \begin{pmatrix} \mathbf{H}(r, \theta', \varphi') \\ \mathbf{E}(r, \theta', \varphi') \end{pmatrix} \sin \theta' d\theta' d\varphi'. \quad (44)$$

This property of the electrodynamic spherical harmonic is very useful for the investigation of electromagnetic wave scattering. In scattering, the boundary condition is the single equation for tangential fields \mathbf{W} . It is easily solved, if the orthonormalization is applied. Some attempts of investigation of scattering in the similar manner as described above have been made in [7].

In the general solution (43) the constants $\mathbf{c}_{1,2}$ determined by initial conditions are explicitly shown. Vectors \mathbf{c}_1 and \mathbf{c}_2 set independent solutions. For instance, if $\mathbf{c}_1 = 0$, then the radial dependence is determined by the function $f^{(2)}(r)$, and vice versa. If $\mathbf{c}_2 = 0$ and $f^{(1)}(r) = h^{(1)}(nkr)$, then the field (43) is the multipole expansion [8]. The amplitude of electric multipole field is equal to $a_E(l, m) = \zeta_1 \mathbf{e}_\theta$, the amplitude of magnetic multipole field is equal to $a_M(l, m) = \zeta_1 \mathbf{e}_\varphi$. So, vector \mathbf{c}_1 can be called vector amplitude of multipole fields.

In further investigations we will study the scattering and multipole expansion of electromagnetic fields in details.

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